

Chapter 9 Eigenvalues

$\{\bar{v}^{(1)}, \dots, \bar{v}^{(k)}\}$ is linearly independent if

$\bar{0} = \alpha_1 \bar{v}^{(1)} + \dots + \alpha_k \bar{v}^{(k)}$ implies all α_i are zeros

If $\{\bar{v}^{(1)}, \dots, \bar{v}^{(n)}\}$ is a set of linearly independent vectors in R^n , then for

any vector $\bar{v} \in R^n$, there exists unique coefficients β_i such that

$$\bar{v} = \beta_1 \bar{v}^{(1)} + \dots + \beta_n \bar{v}^{(n)}$$

If A is a matrix and $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A with

associated eigenvectors $\bar{v}^{(1)}, \dots, \bar{v}^{(k)}$, then $\{\bar{v}^{(1)}, \dots, \bar{v}^{(k)}\}$ is linearly independent.

Orthogonal matrix $Q^T Q = I$

Similar matrices $A = S^{-1} B S$, $A \sim B$

Two similar matrices have the same eigenvalues

Suppose that A is an $n \times n$ symmetric matrix

Then (1) all eigenvalues of A are real

(2) A is similar to a diagonal matrix

(3) all eigenvalues of A are positive if A is positive definite.

Gerschgorin Theorem

Let A be an $n \times n$ matrix and R_i denote the circle in the complex plane

with center a_{ii} and radius $R_i = \{z \in C \mid |z - a_{ii}| \leq \sum_{j=1, j \neq i}^n |a_{ij}|\}$.

The eigenvalues of A are contained within $R = \cup_{i=1}^n R_i$.

§ The power method

1. Suppose a matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$ and the associated

eigenvectors $\bar{v}^{(1)}, \dots, \bar{v}^{(n)}$. Further, $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$

For any vector $\bar{x}$, the following expression is unique

$$\bar{x} = \sum_{i=1}^n \alpha_i \bar{v}^{(i)}$$

$$\text{Then } A^k \bar{x} = \sum_{i=1}^n \alpha_i A^k \bar{v}^{(i)} = \sum_{i=1}^n \alpha_i \lambda_i^k \bar{v}^{(i)} = \lambda_1^k [\alpha_1 \bar{v}^{(1)} + \sum_{i=2}^n \alpha_i \frac{\lambda_i^k}{\lambda_1^k} \bar{v}^{(i)}]$$

$$A^{k+1} \bar{x} = \lambda_1^{k+1} [\alpha_1 \bar{v}^{(1)} + \sum_{i=2}^n \alpha_i \frac{\lambda_i^{k+1}}{\lambda_1^{k+1}} \bar{v}^{(i)}]$$

$$\text{Let } \|\bar{v}^{(1)}\|_{\infty} = 1 = x_p^{(1)}, \text{ Then } \lim_{k \rightarrow \infty} \frac{(A^{k+1} \bar{x})_p}{(A^k \bar{x})_p} = \lambda_1.$$

2. A^{-1} has eigenvalues $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ and the associated eigenvectors

$\bar{x}^{(1)}, \dots, \bar{x}^{(n)}$. $|\lambda_n^{-1}| > \dots > |\lambda_1^{-1}|$. The power method can be used to

calculate λ_n^{-1} and $\bar{v}^{(n)}$.

§ Wielandt deflation

Suppose a matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$ and the associated

eigenvectors $\bar{v}^{(1)}, \dots, \bar{v}^{(n)}$. Further, $|\lambda_1| > |\lambda_2| > \dots > |\lambda_n|$. Let $\bar{x}$ be a vector with

$\bar{x}^T \bar{v}^{(1)} = 1$. Then the matrix $B = A - \lambda_1 \bar{v}^{(1)} \bar{x}^T$ has eigenvalues $0, \lambda_2, \dots, \lambda_n$

with associated eigenvectors $\bar{v}^{(1)}, \bar{w}^{(2)}, \dots, \bar{w}^{(n)}$, where $\bar{v}^{(i)}$ and $\bar{w}^{(i)}$ are related by the equation

$$\bar{v}^{(i)} = (\lambda_i - \lambda_1) \bar{w}^{(i)} + \lambda_1 (\bar{x}^T \bar{w}^{(i)}) \bar{v}^{(1)}, \quad i = 2 \sim n.$$

Remark:

$$A \bar{v}^{(1)} = \lambda_1 \bar{v}^{(1)} \text{ implies } \{a_{i1} \dots a_{in}\} \bar{v}^{(1)} = \lambda_1 v_i^{(1)} \text{ or } \frac{1}{\lambda_1 v_i^{(1)}} \{a_{i1} \dots a_{in}\} \bar{v}^{(1)} = 1$$

$$\text{So, } \bar{x}^T = \frac{1}{\lambda_1 v_i^{(1)}} \{a_{i1} \dots a_{in}\}$$

$$B = A - \lambda_1 \bar{v}^{(1)} \bar{x}^T = A - \lambda_1 \bar{v}^{(1)} \frac{1}{\lambda_1 v_i^{(1)}} \{a_{i1} \dots a_{in}\}$$

The i th row of B is

$$\{b_{i1} \dots b_{in}\} = \{a_{i1} \dots a_{in}\} - \lambda_1 v_i^{(1)} \frac{1}{\lambda_1 v_i^{(1)}} \{a_{i1} \dots a_{in}\}$$

which implies the i th row of B is a zero row.

Since $B\bar{w}^{(2)} = \lambda\bar{w}^{(2)}$, then $w_i^{(2)} = 0$.

Neglecting both the i th row and the i th column yields an $(n-1) \times (n-1)$ matrix B' .

Obtaining the largest eigenvalue $\lambda_1^* (= \lambda_2)$ and its corresponding

eigenvector $\bar{w}^{(1)*}$ of B' . Then $\bar{w}^{(2)}$ is obtained as $\{w_1^{(1)*} \dots 0 \dots w_{n-1}^{(2)*}\}$

and $\bar{v}^{(i)} = (\lambda_i - \lambda_1)\bar{w}^{(i)} + \lambda_1(\bar{x}^T \bar{w}^{(i)})\bar{v}^{(1)}$, $i = 2 \sim n$.

$$\text{Ex. } A = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \text{ has } \lambda_1 = 6, \lambda_2 = 3, \lambda_3 = 1,$$

$$\bar{v}^{(1)} = \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}, \text{ or } \bar{v}^{(1)} = \begin{Bmatrix} -1 \\ 1 \\ -1 \end{Bmatrix}, \bar{v}^{(2)} = \begin{Bmatrix} -2 \\ -1 \\ 1 \end{Bmatrix} \text{ or } \bar{v}^{(2)} = \begin{Bmatrix} 2 \\ 1 \\ -1 \end{Bmatrix}, \bar{v}^{(3)} = \begin{Bmatrix} 0 \\ 1 \\ 1 \end{Bmatrix} \text{ or } \bar{v}^{(3)} = \begin{Bmatrix} 0 \\ -1 \\ -1 \end{Bmatrix}$$

First, take $\bar{v}^{(1)} = \{1, -1, 1\}^T$, $v_1^{(1)} \neq 0$, $\bar{x}^T = \frac{1}{\lambda_1 v_1^{(1)}} \{4, -1, 1\} = \frac{1}{6} \{4, -1, 1\}$

$$B^{(1)} = \begin{bmatrix} 4 & -1 & 1 \\ -1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} - \lambda_1 \bar{v}^{(1)} \bar{x}^T = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 2 & -1 \\ -3 & -1 & 2 \end{bmatrix}$$

Second, neglecting both the first row and the first column of $B^{(1)}$ yields

$$B^{(2)} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

which has the largest eigenvalue $\lambda = 3$ and the associated eigenvector $\bar{w}^{(2)*} = \{1, -1\}^T$.

So, $B^{(1)}$ has the largest eigenvalue $\lambda = 3$ and the associated eigenvector $\bar{w}^{(2)} = \{0, 1, -1\}^T$.

According to $\bar{v}^{(i)} = (\lambda_i - \lambda_1)\bar{w}^{(i)} + \lambda_1(\bar{x}^T \bar{w}^{(i)})\bar{v}^{(1)}$,

we obtain

$$\bar{v}^{(2)} = (3-6)\{0, 1, -1\}^T + 6 \times \frac{1}{6} \{4, -1, 1\} \{0, 1, -1\}^T \{1, -1, 1\}^T$$

$$\bar{v}^{(2)} = \{-2, -1, 1\}^T.$$

$$\text{Third, } \bar{x}^T = \frac{1}{3 \times w_1^{(2)*}} \{2, -1\},$$

$$B^{(3)} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3\bar{w}^{(2)*} \bar{x}^T \quad \text{or}$$

$$B^{(3)} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - 3 \left\{ \begin{matrix} 1 \\ -1 \end{matrix} \right\} \frac{1}{3} \{2 \ -1\} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix},$$

Neglecting the first row and the first column of $B^{(3)}$ yields

$$B^{(4)} = [1],$$

Which has the eigenvalue $\lambda = 1$ and the associated eigenvector $\bar{w}^{(3)*} = \{1\}$. So, $B^{(3)}$ has the largest eigenvalue $\lambda = 1$ and the associated eigenvector $\bar{w}^{(3)} = (0 \ 1)^T$

The eigenvector corresponding $\lambda = 1$ of B is

$$(1-3) \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} + 3 \times \frac{1}{3} \{2 \ -1\} \left\{ \begin{matrix} 0 \\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 1 \\ -1 \end{matrix} \right\} = - \left\{ \begin{matrix} 1 \\ 1 \end{matrix} \right\}$$

$$\bar{v}^{(3)} = \{0, -1, -1\}^T$$

§ Annihilation Technique

Suppose the $n \times n$ matrix A has eigenvalues $\lambda_1, \dots, \lambda_n$ ordered by

$$|\lambda_1| > |\lambda_2| > |\lambda_3| \geq \dots \geq |\lambda_n|$$

with linearly independent eigenvectors $\bar{V}^{(1)}, \dots, \bar{V}^{(n)}$.

Properties:

- The power method applied with $\bar{X}^{(0)} = \beta_2 \bar{V}^{(2)} + \dots + \beta_n \bar{V}^{(n)}$ will obtain λ_2
- For any $\bar{X}$, the power method applied with $\bar{X}^{(0)} = (A - \lambda_1 I) \bar{X}$ will obtain λ_2
- Further, the power method applied with $\bar{X}^{(0)} = (A - \lambda_1 I)(A - \lambda_2 I) \bar{X}$ will obtain λ_3 .

§ Householder's method

The Householder's method is used to reduce a symmetric tridiagonal matrix to a similar matrix that is nearly diagonal.

Define a Householder transformation as

$$P = I - 2\bar{w}\bar{w}^T \in R^{n \times n}, \text{ where } \bar{w}^T \bar{w} = 1. \quad P P^T = I.$$

Given a symmetric matrix A to find $P^{(1)}$ such that $A^{(2)} = P^{(1)} A P^{(1)}$ has

$$a_{j1}^{(2)} = a_{1j}^{(2)} = 0, \quad j = 3, 4, \dots, n.$$

Set $w_1 = 0$, $\bar{w} = \{0 \ w_2 \dots w_n\}^T$, $P^{(1)} = I - 2\bar{w}\bar{w}^T = I - 2\{0 \ w_2 \dots w_n\}^T \{0 \ w_2 \dots w_n\}$

$P^{(1)}\{a_{11} \ a_{21} \ \dots \ a_{n1}\}^T = \{a_{11} \ \alpha \ 0 \ 0 \ \dots \ 0\}^T$, α is unknown.

$\hat{P}^{(1)} = I - 2\bar{w}\bar{w}^T = I - 2\{w_2 \dots w_n\}^T \{w_2 \dots w_n\}$, $\hat{P}^{(1)}\{a_{21} \ \dots \ a_{n1}\}^T = \{\alpha \ 0 \ 0 \ \dots \ 0\}^T$

Let $\{w_2 \dots w_n\}\{a_{21} \ \dots \ a_{n1}\}^T = r$, $\{\alpha \ 0 \ 0 \ \dots \ 0\}^T = \{a_{21} - 2rw_2 \ \dots \ a_{n1} - 2rw_n\}$

implies $2rw_2 = a_{21} - \alpha$, $2rw_j = a_{j1}$, $j=3, \dots, n$.

Based on $\sum_{j=2}^n w_j^2 = 1$, $4r^2 = (a_{21} - \alpha)^2 + a_{j1}^2 = \sum_{j=2}^n a_{j1}^2 - 2\alpha a_{21} + \alpha^2$

$\alpha^2 = \{\alpha \ 0 \ 0 \ \dots \ 0\}\{\alpha \ 0 \ 0 \ \dots \ 0\}^T = \{a_{21} \ \dots \ a_{n1}\}^T \hat{P}^{(1)T} \hat{P}^{(1)}\{a_{21} \ \dots \ a_{n1}\}^T$

$\alpha^2 = \{a_{21} \ \dots \ a_{n1}\}\{a_{21} \ \dots \ a_{n1}\}^T = \sum_{j=2}^n a_{j1}^2$

implies $2r^2 = \alpha^2 \sum_{j=2}^n a_{j1}^2 - \alpha a_{21} = \alpha^2 - \alpha a_{21}$

Choose $\alpha = -\text{sgn}(a_{21})(\sum_{j=2}^n a_{j1}^2)^{1/2}$, then $r = (\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha a_{21})^{1/2}$,

$w_1 = 0$, $w_2 = \frac{a_{21} - \alpha}{2r}$, $w_j = \frac{a_{j1}}{2r}$, $j=3, \dots, n$.

$$A^{(2)} = P^{(1)}AP^{(1)} = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & 0 & 0 \\ a_{21}^{(2)} & \cdot & \cdot & a_{2n}^{(2)} \\ 0 & \cdot & \cdot & \cdot \\ 0 & a_{n2}^{(2)} & \cdot & a_{nn}^{(2)} \end{bmatrix}$$

Next, $\alpha = -\text{sgn}(a_{k+1,k}^{(k)})(\sum_{j=k+1}^n a_{jk}^{(k)2})^{1/2}$, $r = (\frac{1}{2}\alpha^2 - \frac{1}{2}\alpha a_{k+1,k}^{(k)})^{1/2}$,

$w_1^{(k)} = w_2^{(k)} = \dots = w_k^{(k)} = 0$, $w_{k+1}^{(k)} = \frac{a_{k+1,k}^{(k)} - \alpha}{2r}$, $w_j^{(k)} = \frac{a_{jk}^{(k)}}{2r}$, $j=k+2, \dots, n$.

$A^{(k+1)} = P^{(k)}A^{(k)}P^{(k)}$.

Continuing the process yields $A^{(n-1)} = P^{(n-2)} \dots P^{(1)}AP^{(1)} \dots P^{(n-2)}$.

§ QR Factorization

If A is an $m \times n$ matrix of rank n , then A can be factored into a product QR , where Q is an $m \times n$ matrix with orthonormal columns and R is an $n \times n$ matrix that is upper triangular and invertible.

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$$A = [\bar{a}_1 \dots \bar{a}_n] = [\bar{q}_1 \dots \bar{q}_n] \begin{bmatrix} r_{11} & \dots & \dots & r_{1n} \\ 0 & r_{22} & \dots & r_{2n} \\ 0 & 0 & \dots & \cdot \\ 0 & 0 \dots & 0 & r_{nn} \end{bmatrix},$$

$$\bar{a}_1 = r_{11}\bar{q}_1, \quad \bar{a}_2 = r_{12}\bar{q}_1 + r_{22}\bar{q}_2, \quad \bar{a}_3 = r_{13}\bar{q}_1 + r_{23}\bar{q}_2 + r_{33}\bar{q}_3, \dots$$

$$\bar{a}_i = r_{1i}\bar{q}_1 + r_{2i}\bar{q}_2 + r_{3i}\bar{q}_3 + \dots + r_{ii}\bar{q}_i, \quad \bar{q}_i^T \bar{q}_j = \delta_{ij}$$

$$\|\bar{a}_1\| = r_{11}\|\bar{q}_1\| = r_{11} \quad \text{implies} \quad \bar{q}_1 = \bar{a}_1 / r_{11}$$

$$r_{12} = \bar{q}_1^T \bar{a}_2, \quad \bar{a}_2 - r_{12}\bar{q}_1 = r_{22}\bar{q}_2, \quad r_{22} = \|\bar{a}_2 - r_{12}\bar{q}_1\|, \dots$$

$$r_{ii}\bar{q}_i = \bar{a}_i - r_{1i}\bar{q}_1 - r_{2i}\bar{q}_2 - r_{3i}\bar{q}_3 - \dots, \quad \bar{q}_j^T \bar{a}_i = r_{ji}, \quad 1 \leq j < i,$$

$$r_{ii} = \|\bar{a}_i - r_{1i}\bar{q}_1 - r_{2i}\bar{q}_2 - r_{3i}\bar{q}_3 - \dots\|, \quad \bar{q}_i = (\bar{a}_i - r_{1i}\bar{q}_1 - r_{2i}\bar{q}_2 - r_{3i}\bar{q}_3 - \dots) / r_{ii}.$$

$$\text{Let } A^{(2)} = RQ, \quad A^{(2)} = Q^{-1}AQ = Q^T AQ$$

HW

1. Use the power method to estimate all eigenvalues of

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$